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THE FINITE ELEMENT AND BOUNDARY ELEMENT METHODS IN PROBLEMS OF THE DYNAMICS OF ELASTIC VESSELS WITH A LIQUID†

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A version of a method of calculating the natural longitudinal vibrations of an elastic axisymmetric vessel, partially filled with liquid, is described. The basis of the method is a combination of boundary elements for the liquid and finite elements for the elastic vessel. Examples which illustrate the effectiveness of the method are given. © 2004 Elsevier Ltd. All rights reserved.

The fundamental formulations of the problems of the dynamic interaction of a thin-walled elastic vessel with a liquid filling it have been discussed previously in [1, 2]. The proposed algorithms for calculating the dynamic characteristics of the longitudinal vibrations of a vessel, designed for use on a computer, to a certain extent have solved part of the problem for vessels of simple shapes [3, 4]. Their further development requires progress in computational techniques [5, 6].

The development of constructive rocket-carrier schemes requires improvements in the calculation algorithms. The existing NASTRAN-type software packages do not always enable an acceptable solution to be obtained.

The proposed method is based on a more flexible description of the geometry of the vessel using effective finite elements, which enables the results of calculations to be refined using well-known methods, and enables vessels of complex geometry to be calculated taking into account their internal structural features.

1. FORMULATION OF THE PROBLEM

Consider a thin-walled elastic vessel, consisting of shells of revolution, reinforced by ribs, partially filled with an incompressible inviscid liquid. The motion of the liquid is assumed to be potential; we will neglect the energy of wave formation on the free surface of the liquid.

We will connect with the vessel a cylindrical system of coordinates z, r, ϑ (Fig. 1). The external strengthening of the vessel is carried out by means of ribs using ring brackets, which possess distributed stiffness in the longitudinal and radial directions, and also angular stiffness. In the theoretical scheme, the rigid couplings are represented by fairly large stiffnesses in the required directions. Inside the vessel there may be rod structures, which do not affect the flow of the liquid. Additional masses can be attached to the ribs. We will consider small longitudinal harmonic vibrations of the vessel with an angular frequency ω (all the functions considered have a time factor $\exp(i\omega t)$).

This structure with the liquid represents a system with an infinite number of degrees of freedom. To describe it approximately by a system with a finite number of degrees of freedom we will use two well-known methods below: the finite element method, to represent the elastic elements of the vessel and the boundary element method, to stimulate the motions of the liquid in the vessel.

When formulating the problem for shells using the finite element method there are different possibilities for choosing the approximating functions of the shape. The requirements imposed on these functions are well known [7]: these functions must re-establish the displacement of an element as a rigid whole, and when reducing the dimensions of the element they must describe the state of constant deformation. The problems involved in constructing a curvilinear element possessing such properties are well known [8, 9]. A separate problem is to describe the initial geometry of the object in a form

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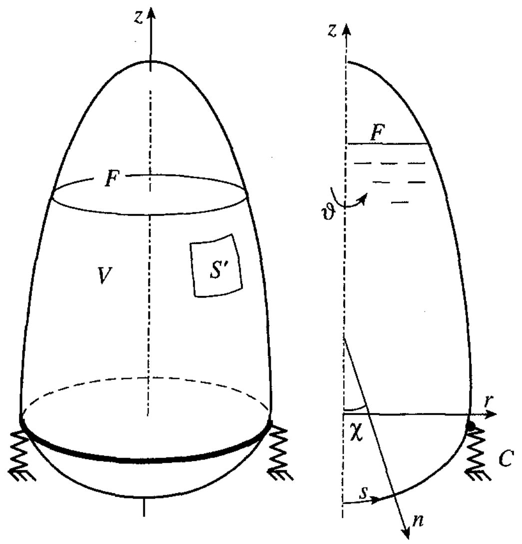

Fig. 1

that is convenient for effective use on a computer, in particular for simplifying the procedure for increasing the number of finite elements.

The boundary element method imposes particular requirements on the functions which approximate the boundary of the liquid volume, and on the form functions describing the displacements of this boundary.

2. DESCRIPTION OF THE MERIDIAN OF THE AXISYMMETRIC VESSEL

The parametric equations of the meridian – a plane curve, has the simplest form when the parameter is the length of the arc of the curve s. In this description one can approximately construct part of the curve using information at neighbouring nodes m and $m + 1$ as follows [10].

We will write the radius vector of part of the curve in the form

$$\mathbf{r}(s) = x_k(s)\mathbf{i}_k, \quad k = 1, 2$$

where $\mathbf{i}_k$ are the unit vectors of the global system of coordinates.

The derivative of the vector $\mathbf{r}$ with respect to the arc has unit length. We will put $dx_1/ds = \cos\chi$ and $dx_2/ds = \sin\chi$, where χ is the angle of inclination of the normal to the curve. We will specify the variation of the angle χ along the length of the curve in the form

$$\chi(s) = \left(1 - \frac{s}{l}\right)\chi_m + \frac{s}{l}\chi_{m+1} + B\left(1 - \frac{s}{l}\right)\frac{s}{l} \tag{2.1}$$

where χ_m, χ_{m+1} are the angles at the points m and $m + 1$ respectively, and B and l are an unknown coefficient and an unknown length of the curve between the points, to be determined. For unknown B and l we obtain the following system of two non-linear equations

$$x_{1m+1} = x_{1m} + \int_0^l \cos\left[\left(1 - \frac{s}{l}\right)\chi_m + \frac{s}{l}\chi_{m+1} + B\left(1 - \frac{s}{l}\right)\frac{s}{l}\right]ds$$

$$x_{2m+1} = x_{2m} + \int_0^l \sin\left[\left(1 - \frac{s}{l}\right)\chi_m + \frac{s}{l}\chi_{m+1} + B\left(1 - \frac{s}{l}\right)\frac{s}{l}\right]ds$$

We will use Newton's method to solve this system.

Since in approximation (2.1) we have established the possibility of describing the constant curvature of the curve, then, using the algorithm described, we can establish the element of the circle of any aperture angle with a high degree of accuracy, and, in particular, the rectilinear part.

Hence, the curvilinear contour of the vessel can be determined on a computer using the coordinates of individual points and the directions of the normals to the contour at these points. The length of the curvilinear parts are calculated in the course of the approximation.

3. DESCRIPTION OF THE MOTION OF THE LIQUID USING THE FINITE ELEMENT METHOD

The above assumptions regarding the motions of the liquid in a volume V are fundamental for introducing the displacement potential of the liquid φ [11]. The potential φ satisfies the boundary-value problem

$$\Delta\varphi = 0 \text{ in } V; \quad \varphi = 0 \text{ on } F, \ \frac{\partial\varphi}{\partial n} = \mathbf{U}\cdot\mathbf{n} \text{ on } S'$$

where $\mathbf{U}$ is the vector of the elastic displacement of the wall of the vessel, $\mathbf{n}$ is the vector of the outward normal to the wall, F is the free surface of the liquid and S' is the wetted surface of the vessel.

The formulate the problem of the vibrations of the vessel with the liquid we must obtain the relation between the boundary values of φ and $\partial\varphi/\partial n$. This relation between the corresponding values at a finite number of points of the wetted contour of the vessel is established using the direct boundary element method.

The realization of the direct boundary element method is based on the use of Green's integral theorem for two functions that are harmonic in the region V: the required potential φ and the fundamental solution of Laplace's equation for axisymmetric flow of the liquid. The fundamental solution is specified in such a form that the solution for the displacement potential obtained by the direct boundary element method exactly satisfies the condition on the free surface of the liquid. In the problem in question, the fundamental solution has the form

$$\Phi(x, x^{(i)}) = G(x, x^{(i)}) - \bar{G}(x, x^{(i)})$$

$$G(x, x^{(i)}) = \frac{1}{\pi r_1}K\left(1 - \frac{r^2}{r_1^2}\right), \quad \bar{G}(x, x^{(i)}) = \frac{1}{\pi r_1^*}K\left(1 - \frac{r^{*2}}{r_1^{*2}}\right)$$

where $K(\ldots)$ is the complete elliptic integral of the first kind, x is the actual point on the boundary of the region, $x^{(i)}$ is the collocation point on the boundary, r is the distance between the points x and $x^{(i)}$, r_1 is the distance between the point x and point $x'^{(i)}$, symmetrical to the point $x^{(i)}$ relative to the axis of symmetry of the region occupied by the liquid, and r^* and r_1^* are the distances between the point x and the points symmetrical to the point $x^{(i)}$ and $x'^{(i)}$ with respect to the liquid surface.

In green's formula the integration is carried out over the wetted surface. The scheme for obtaining the resolvent relations by the direct boundary element method are standard when using boundary elements with a constant distribution of the unknown boundary functions. In this case the collocation points were taken in the middle of each element, and to match the boundary element and finite element schemes, two boundary elements were taken on each finite element. The singular integrals are elevated using a special quadrature formula containing logarithms. As a result of solving the system of equations by the direct finite-element method for the unknown nodal values of the potential φ, the kinetic energy function of the liquid (the kinetic energy without the factor ω^2 – the square of the angular frequency) is reduced to a quadratic form of the unknown nodal values of $\partial\varphi/\partial\mathbf{n}$, which in turn are expressed in terms of the nodal values of the generalized displacements of the finite element scheme of the vessel.

When using appropriate form functions of the wetted finite element of the shell as $\partial\varphi/\partial\mathbf{n}$, the problem of matching the finite element method scheme and boundary element method scheme does not arise.

The use of this approach enables us to write the kinetic energy function of the liquid in the form of a quadratic form of the generalized nodal unknowns of the finite element scheme

$$T = \frac{1}{2}\mathbf{Y}^T\mathbf{T}_f\mathbf{Y}$$

where $\mathbf{T}_f$ is the matrix of the added masses of liquid, and $\mathbf{Y}$ is the vector of the generalized nodal unknowns of the finite element scheme.

4. THE FINITE ELEMENT DESCRIPTION OF THE SHELL

In the problem considered, the form of the shells of the vessel before and after its deformation is defined by a plane curve – the meridian. The construction of the finite element of such a vessel is equivalent to the construction of an element for a plane curvilinear rod. It is proposed to obtain the form functions of the finite element of a plane rod as a result of integrating specified changes in the angles of rotation of the normal to the rod and the deformation along the rod [12]. The relation between the deformation ε, the angle of rotation of the normal to the element $\Delta\chi$ and the arbitrary global components of the displacement vector U_1 and U_2 along the arcs of the contour of the vessel has the form

$$dU_k/ds = \varepsilon dx_k/ds + (-1)^k \Delta\chi dx_{3-k}/ds, \quad k = 1, 2 \tag{4.1}$$

These relations also retain their form for any other parameterization of the axial length of the rod. They can form the basis for constructing the form functions.

Different versions of the representation of the deformation and the angle of rotation of the finite element are possible. We will assume here that the unknown deformation ε is constant on the element. We will specify the angle of rotation of the normal to the element when it is deformed in the form

$$\Delta\chi(s) = \Delta\chi_j(1 - L(s)) + \Delta\chi_{j+1}L(s) + B(1 - L(s))L(s); \quad L(s) = \frac{s - s_j}{s_{j+1} - s_j} \tag{4.2}$$

where **j** and **j** + 1 are the nodes of the finite element, $\Delta\chi_j$, $\Delta\chi_{j+1}$ are the angles of rotation of the normal to the element at these nodes, s_{j+1} and s_j are the coordinates of the nodes of the element along the arc, and B is a constant, to be determined together with ε.

We will use the following notation: U_{1j} and U_{2j} are the components of the displacement of the node j and U_{1j+1} and U_{2j+1} are the components of the displacement of the node $j + 1$.

Integrating (4.1) using relation (4.2) we obtain

$$U_k(s) = U_{kj} + \varepsilon[x_k(s) - x_{kj}] + (-1)^k \int^s \Delta\chi(y)\frac{dx_{3-k}}{dy}dy, \quad k = 1, 2$$

The constants B and ε are found from the conditions at the node $j + 1$

$$U_k(s_{j+1}) = U_{kj+1}, \quad k = 1, 2$$

and are expressed in terms of the generalized nodal displacements of the finite element.

In the special case of a straight rod, the approach described leads to well-known form functions [7].

We will give an example of the effectiveness of the finite element constructed. Consider a plane rod with an axial line in the form of a semicircle cantilevered in a vertical wall and loaded at the free end with loads in its plane: a concentrated moment and concentrated forces. For equal dimensionless loads, the maximum error occurs in determining the horizontal displacement of the free end. This amounts to 0.6% for one finite element compared with the exact solution.

This method of finding the form functions is also used when constructing a curvilinear finite element of a shell of the vessel.

The expressions for the kinetic and potential energies of the dry vessel can be written in the form

$$T = \frac{1}{2}\mathbf{Y}^T\mathbf{T}_\zeta\mathbf{Y}, \quad \Pi = \frac{1}{2}\mathbf{Y}^T\mathbf{\Pi}_\zeta\mathbf{Y}$$

where $\mathbf{T}_\zeta$ is the matrix of the masses of the vessel and its ribs, and $\mathbf{\Pi}_\zeta$ is the matrix of the stiffness of the elastic elements of the vessel, including the shells, the ribs and the distributed supports.

The problem of the natural longitudinal vibrations of the vessel with the liquid is formulated as a generalized eigenvalue problem

$$(\mathbf{\Pi}_\zeta - \omega^2(\mathbf{T}_\zeta + \mathbf{T}_f))\mathbf{Y} = 0$$

This problem is solved by iterations in a subspace.

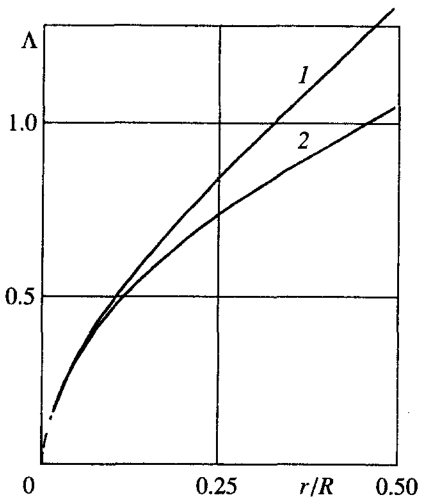

Fig. 2

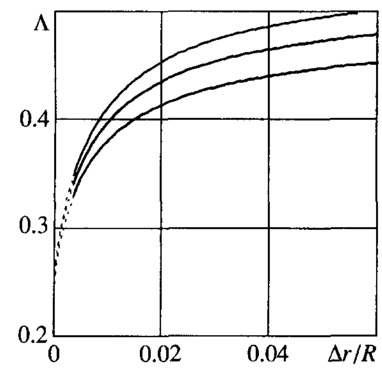

Fig. 3

5. THE RESULTS OF CALCULATIONS

The dynamic characteristics of a spherical vessel with a liquid for average fillings have been analysed by many methods (see, for example, [3–6]). When the level to which the vessel is filled is increased the radius of the liquid table r approaches zero. The frequency of the lowest one, defined by the volume expansion of the vessel, for small r is proportional to $\sqrt{r}$. This asymptotic form of the lowest tone also appears in calculations using the algorithm considered.

In Fig. 2 we show graphs of the eigenvalue as a function of the radius of the liquid table for the first elastic tone of a free spherical vessel (curve 1) and for a spherical vessel clamped in the longitudinal direction along the equator (curve 2). The vessel is filled with water. The parameters of the vessel are as follows: $\rho = 2700$ kg/m^3 is the density of its material, $E = 7 \times 10^{10}$ N/m^2 is its modulus of elasticity, $\nu = 0.3$ is Poisson's ratio, $R = 1$ m is the radius of the generatrix, and $h = 0.005$ m is the wall thickness. The eigenvalue Λ is related to the angular frequency ω by the formula

$$\Lambda = \sqrt{\frac{\rho R^3(1-\nu^2)}{Eh}}\omega \tag{5.1}$$

A toroidal vessel, free or clamped along the equator, has a low frequency of elastic vibrations compared with the remaining frequencies. Thus, for a dry circular toroidal vessel of inner radius 1 m and outer radius 3 m made of dural 10 mm thick, clamped along the inner equator, the first three frequencies are 11 Hz, 167 Hz and 147 Hz. For a torus, half filled with water, the frequencies are 4.5 Hz, 60 Hz and 68Hz. The reason for the occurrence of a low first frequency is explained by the geometry of the torus-shaped shell. A stressed state, close to a pure moment state, occurs in the region of the lines of zero Gaussian curvature of the middle surface of the shell. In this form of vibration, the inner and outer parts of the toroidal vessel, separated by lines of zero Gaussian curvature, move along the axes of symmetry of the vessel as practically underformed elements.

The vibration tones, determined by the volume expansion of the vessel for large fillings, as also in the case of a sphere considered above, are also of low frequency, but this is not related to the way in which the vessel is supported.

The effect of degeneracy of the vibration tone, determined by the volume expansion of the toroidal vessel, manifests itself most clearly when the vessel is fixed along both equators, when the low-frequency tone described above is not present. This effect is also revealed by calculation in all other cases of attachment.

In Fig. 3 we show a graph of the eigenvalue (5.1) of the degenerate vibration tone as a function of the width of the ring of the free surface of the liquid Δr for a toroidal vessel filled with water (the upper curve is the free vessel, the middle curve is a vessel clamped in a longitudinal direction along the inner equator, and the lower curve is a vessel clamped in a longitudinal direction along both equators). The characteristics of the toroidal vessel were as follows: radius of the outer equator 3 m, radius of the inner equator 1 m, wall thickness $h = 0.01$ m, and the density of its material, the modulus of elasticity and

Table 1

Frequency (H2)	Experiment	Calculation			
		$N = 10$	20	30	40
f_1	216	213	212.6	212.5	212.4
f_2	305	325.8	324.2	323.8	323.8
f_3	370	401.1	399.8	399.3	399.1

Poisson's ratio are the same as in the previous case of a spherical vessel. Figure 3 does not show the tones related to the geometrical features of the toroidal vessel; they are practically independent of the width of the ring.

The number of finite elements in the calculations did not exceed 20 on a semicircle of the generatrix of the vessel.

We will give some results of a comparison of the calculations with experimental data. We calculated a vessel of hemispherical form, completely filled with water, rigidly constrained along an equator. The characteristics of the vessel were as follows: $\rho = 1180$ kg.m^{-3} is the density of the material of the shell, $E = 4.016 \times 10^9$ N.m^{-2} is the modulus of elasticity of the material, $\nu = 0.4$ is Poisson's ratio, $R = 0.133$ m is the radius of the generatrix of the vessel and $h = 0.0007$ m is the wall thickness. The table shows the frequencies of axisymmetric vibrations of a hemispherical vessel with the same parameters, obtained experimentally [13] and by calculation using the above algorithm (N is the number of finite elements).

A comparison of the values of the first eigenvalue obtained for $N = 10$ and obtained by solving the problem for a clamped moment shell using Legendre polynomials for several different conditions on the surface of the liquid (a floating cover) [14], gives a difference of 0.8%.

We wish to dedicate this paper to the 80th birthday of Corresponding member of the Russian Academy of Sciences E. I. Grigolyuk.

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